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Quasiperiodic structure of the stochastic web map

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The q-fold web map is extended to a volume-preserving map in q dimensions. The qth iterate of this map is invariant under translations by 2π in any of the coordinate directions. As a consequence, well-behaved invariant sets that are periodic in $\mathbb{R}^q$ produce quasiperiodic sets when restricted to a two-dimensional, irrationally placed subspace identified with the original phase space. A quasilattice of fixed points is constructed in this manner. It is conjectured that the stochastic web, considered as the restriction of a q-dimensional periodic set , is itself quasiperiodic.

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The stochastic web map [1–3],

$$M_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 2\pi/q & \sin 2\pi/q \\ -\sin 2\pi/q & \cos 2\pi/q \end{pmatrix}\begin{pmatrix} x \\ y + a\sin x \end{pmatrix} \tag{1}$$

is the Poincaré map of a one-dimensional harmonic oscillator that is given instantaneous kicks q times per natural period. One of its remarkarkable properties is the dynamical generation of ordered structure [2]: a single chaotic orbit of M_2 is capable of tracing out a weblike region, the *stochastic web*, which extends throughout the phase plane and is endowed, at least approximately, with the long-range translational and orientational order associated with a crystal ($q = 2,3,4,6$) or quasicrystal ($q = 5,7,8,\ldots$).

In the simplest quasicrystalline case, $q = 5$, numerical investigations [2,3] have revealed strong connections with the best known of all two-dimensional quasicrystals, the Penrose tiling of the plane by two types of rhombuses [4]. For example, a Penrose tiling may be superimposed on the phase portrait of M_2 in such a way that the local features of the latter provide an approximate decoration of the tiles [2]. Also intriguing are numerical calculations [3] of the Fourier transform of a large circular portion of the stochastic web generated by a single chaotic orbit. The result is reminiscent of the pointlike diffraction pattern of the Ammann quasilattice [5–8] associated with the Penrose tiling.

For $a \ll 1$, the qth iterate map M_2^q is asymptotically close to a Hamiltonian flow, with

$$M_2^q\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + a\begin{pmatrix} \dfrac{\partial H_2}{\partial y} \\ -\dfrac{\partial H_2}{\partial x} \end{pmatrix} + O(a^2), \tag{2}$$

$$H_2(x,y) = \sum_{k=0}^{q-1}\cos\left(x\cos\frac{2k\pi}{q} + y\sin\frac{2k\pi}{q}\right). \tag{3}$$

The Hamiltonian H_2 is obviously *quasiperiodic*, in the sense that it has a Fourier expansion with at most a countable number of wave vectors. As we shall see below, so is the characteristic function of the level set

$$\{(x,y):\ |H_2(x,y) - E| < \Delta E\}. \tag{4}$$

Numerical investigations [2,3] indicate that the region of the xy plane filled out by a typical chaotic orbit is a good approximation to the infinite connected component of some level set of H_2. The approximation is found to improve with decreasing a, with an error of order a^2. It is in this sense of a first-order perturbative approximation that the notion of dynamical generation of quasiperiodic structure was introduced by Zaslavsky and collaborators [2]. With the help of higher-order Hamiltonians introduced in [9], the concept can be broadened to arbitrary order in a, giving a good approximation of the web by level sets for moderately large values of the parameter, say $a < 0.6$.

The present work is aimed at sharpening the above results. We shall see that there are certain sets of fixed points of M_2^q that are not just nth-order approximations to quasiperiodic structures, but are themselves quasiperiodic. This result will be seen to follow from the periodicity property of a q-dimensional map that is an extension of M_2^q. Similar reasoning leads to a plausible conjecture that the stochastic web itself is quasiperiodic.

We begin by reviewing one of the simplest ways [10,11] of generating a quasiperiodic function in d dimensions. Consider a real-valued function $f(\zeta)$ over $\mathbb{R}^q, q > d$, which is periodic with respect to 2π translations in any of the q coordinate directions:

$$f(\zeta + 2\pi\nu) = f(\zeta), \quad \nu \in \mathbb{Z}^q,$$

and suppose that S is a d-dimensional subspace of $\mathbb{R}^q$ that is irrationally placed; that is, S contains no integer lattice points $2\pi\nu, \nu \in \mathbb{Z}^q$ other than the origin. Further, suppose that f is sufficiently well behaved that it has a Fourier series expansion

$$f(\zeta) = \sum_{\kappa\in\mathbb{Z}^q} c_\kappa e^{i\kappa\zeta}. \tag{5}$$

If the restriction of (5) to $\zeta \in S$ is well defined, it has the form

$$f_S(\zeta) = \sum_{\kappa\in\mathbb{Z}^q} c_\kappa e^{i\phi_S(\kappa)\zeta}, \quad \zeta \in S, \tag{6}$$

where ϕ_S is the orthogonal projector onto S. The function f_S is thus, by definition, quasiperiodic over S. This idea is the underlying principle of the direct projection method [10–14] of constructing quasiperiodic tilings in one-, two-, and three-dimensional spaces, a useful tool for modeling the diffraction patterns of physical quasicrystals.

To utilize the method just described, we need to work within a mathematical franework that goes beyond continuous functions. The space of tempered distributions (class S' generalized functions [15]) is well suited to our needs (see [8] for an alternative choice). These are defined as linear functionals on a function space, and hence are not necessarily meaningful at an individual point. This could pose a problem: even if the series (5) converges in the distributional sense, the restriction (6) may fail to exist. This is something that will have to be checked in each individual case.

Before turning to the web map, let us see how the quasiperiodicity of the Hamiltonian H_2 and its level sets can be understood in terms of a q-dimensional embedding [16]. Clearly H_2 is the restriction of the q-dimensional periodic function

$$H_q(\zeta) = \sum_{k=0}^{q-1} \cos\zeta_k \tag{7}$$

to the two-dimensional subspace S represented parametrically by

$$\zeta = \xi(\mathbf{x}) \equiv x\,\omega^{(1)} + y\,\omega^{(2)}, \tag{8}$$

where $\mathbf{x} \equiv (x, y)$, and $\omega^{(1)}$ and $\omega^{(2)}$ are mutually orthogonal vectors in S, with components

$$\omega_k^{(1)} = \cos\frac{2\pi k}{q}, \quad \omega_k^{(2)} = \sin\frac{2\pi k}{q}, \quad k = 0, 1, \ldots, q-1. \tag{9}$$

By the same token, the level sets (4) of H_2 are restrictions to S of the periodic level sets of H_q with the same E and ΔE. For $q \notin \{1, 2, 3, 4, 6\}$, the subspace S is irrationally placed, and hence the Hamiltonian H_2 and the characteristic functions of its level sets are quasiperiodic functions whose Fourier coefficients can be obtained from a knowledge of their q-dimensional extensions in the fundamental hypercube,

$$\{\zeta : \ |\zeta_i| \le \pi, \ i = 0, 1, \ldots, q-1\}.$$

The key insight that allows us to associate quasiperiodic structures with the web map is the following: the web map M_2 is the restriction to the irrationally placed subspace S of a map M_q on $\mathbb{R}^q$ whose qth iterate is invariant under 2π translations in any of the coordinate directions. Specifically, let P_q be the cyclic permutation map that takes $(\zeta_0, \zeta_1, \ldots, \zeta_{q-1})$ into $(\zeta_1, \ldots, \zeta_{q-1}, \zeta_0)$. The two-dimensional subspace S, which we identify with the xy plane via (8), is left invariant by P_q, and the induced map on S is simply a clockwise rotation by $2\pi/q$:

$$P_q(\xi(\mathbf{x})) = \xi\left(x\cos\frac{2\pi}{q} + y\sin\frac{2\pi}{q},\right.$$
$$\left. -x\sin\frac{2\pi}{q} + y\cos\frac{2\pi}{q}\right). \tag{10}$$

Now we are in a position to define the q-dimensional extension of M_2,

$$M_q(\zeta) = P_q\left(\zeta + a\sin\zeta_0\,\omega^{(2)}\right). \tag{11}$$

Clearly,

$$M_q(\xi(\mathbf{x})) = \xi(M_2(\mathbf{x})), \tag{12}$$

and, for $\nu \in \mathbb{Z}^q$,

$$M_q(\zeta + 2\pi\nu) = M_q(\zeta) + 2\pi P_q(\nu), \tag{13}$$

so that, iterating q times, we get the translational invariance

$$M_q^q(\zeta + 2\pi\nu) = M_q^q(\zeta) + 2\pi\nu. \quad (14)$$

The subspace S, which we have identified as the original two-dimensional phase space, is only one of an infinite set of parallel hyperplanes left invariant by M_q^q. To see this, we write, for $\sigma \in S$,

$$M_q(\sigma + \eta) = P_q(\sigma + a\sin(\sigma_0 + \eta_0)\,\omega^{(2)}) + P_q(\eta). \quad (15)$$

It is clear from (15) that M_q^q leaves invariant each hyperplane

$$S_\eta = \eta + S = \{\sigma + \eta : \ \sigma \in S\},$$

where, without loss of generality, we assume that η is in $S_\perp$, the $(q-2)$-dimensional orthogonal complement of S. The map induced on S_η, parametrized by $\mathbf{x}$, is the two-dimensional area-preserving map

$$\tilde{M}_\eta^{(q)} = \tilde{M}_{\eta_{q-1}} \circ \tilde{M}_{\eta_{q-2}} \circ \cdots \circ \tilde{M}_{\eta_0}, \quad (16)$$

where

$$\tilde{M}_z \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 2\pi/q & \sin 2\pi/q \\ -\sin 2\pi/q & \cos 2\pi/q \end{pmatrix} \begin{pmatrix} x \\ y + a\sin(x+z) \end{pmatrix}. \quad (17)$$

For asymptotically small a, application of M_q^q approaches a continuous flow:

$$\tilde{M}_q(\xi(\mathbf{x}) + \eta) = \xi(\mathbf{x}) + \eta + a\left(\frac{\partial H_\eta}{\partial y}(\mathbf{x})\,\omega^{(1)} - \frac{\partial H_\eta}{\partial x}(\mathbf{x})\,\omega^{(2)}\right) + O(a^2), \quad (18)$$

where

$$H_\eta(\mathbf{x}) = \sum_{k=0}^{q-1} \cos[\eta + \xi(\mathbf{x})]_k. \quad (19)$$

We see that the flow, while volume preserving in q dimensions, is intrinsically two dimensional. Only x and y are dynamical variables, with the vectors in $S_\perp$ serving as parameters of the dynamical system.

An interesting example of a quasiperiodic structure generated by the web map is what we shall call *quasicenters*, namely fixed points $\mathbf{x}^*$ of M_2^q such that the qth iterate map relative to $\mathbf{x}^*$ is close to M_2^q in the sense that

$$|\tilde{M}_{\xi(\mathbf{x}^*)}^{(q)}(\mathbf{x}) - M_2^q(\mathbf{x})|/a$$

is uniformly bounded by a small positive number. From (16) and (17), it is clear that this can be achieved if all components of $\xi(\mathbf{x}^*)$ are close to integer multiples of 2π. That this is also a necessary condition can also be shown. We therefore define the set of *quasicenters of tolerance* ϵ to be the restriction to S of the q-dimensional periodic set specified by the fixed point condition

$$M_q^q(\zeta^*) = \zeta^*, \quad (20)$$

supplemented by a cutoff condition that restricts ζ^* to be close to some lattice point $2\pi\nu$, $\nu \in \mathbb{Z}^q$.

Observe that Eq. (20), when restricted to the fundamental hypercube, determines fixed points $\mathbf{x}^*(\eta)$ in each of the hyperplanes S_η, $\eta \in S_\perp$. Since the origin is an isolated fixed point in S, and since the fixed-point equation (20) is analytic in all its variables, there is a unique solution for $|\eta| < \epsilon$, for ϵ chosen sufficiently small. The set defined by (20) and $|\eta| < \epsilon$ is thus part of a smooth, dimension-$(q-2)$ manifold Ξ that intersects S in the origin. Introducing (x, y) coordinates in S, the generalized function $f_0(\zeta)$, defined by

$$f_0(\xi(\mathbf{x}) + \eta) = \delta^{(2)}(\mathbf{x} - \mathbf{x}^*(\eta))\,\Theta(\epsilon - |\eta|), \quad (21)$$

for arbitrary $\xi(\mathbf{x}) \in S$ and $\eta \in S_\perp$, is well defined in $\mathbb{R}^q$ with support on Ξ. It is readily extended to a periodic generalized function

$$f(\zeta) = \sum_{\nu \in \mathbb{Z}^q} f_0(\zeta + 2\pi\nu), \quad (22)$$

whose restriction to S, $f_S(\mathbf{x})$, consists of a quasiperiodic sum of δ functions , each with unit weight. The associated diffraction pattern is obtained by expanding $f(\zeta)$ in a q-dimensional Fourier series, with coefficients

$$\tilde{f}_\kappa = (2\pi)^{-q} \int d^{q-2}\eta\,\Theta(\epsilon - |\eta|)\,\exp\left[-i\kappa(\eta + \xi(\mathbf{x}^*(\eta)))\right], \quad \kappa \in \mathbb{Z}^q. \quad (23)$$

Restriction to S then gives us a quasiperiodic series of the type (6).

We now come to a very difficult and subtle question: Is the stochastic web itself quasiperiodic in the plane? To even start to answer this, we must define precisely what we mean by the web. Intuitively, it is useful to think of the plane as a two-dimensional "foam," consisting of the stochastic web and its complement, a countably infinite set of disjoint "bubbles." Each "bubble" is a simply connected region bounded by an M_2^q-invariant curve that is *maximal* in the sense that it is not contained in the interior of any other simple, closed, invariant curve. We shall refer to such curves simply as boundary curves. Numerical explorations [2,3] indicate that for any positive a sufficiently small that the origin is a stable fixed point, a boundary curve surrounding the origin indeed exists. We define the stochastic web to be the infinitely extended region exterior to the set (assumed to be nonempty) of all boundary curves. Note that with this definition the contents of the web are not restricted to chaotic orbits. Also to be found within its boundaries are periodic orbits, cantori, and islands invariant under M_2^{nq}, $n > 1$.

The question about the quasiperiodicity of the stochastic web can now be formulated as follows. Let $\chi_{\text{web}}(\mathbf{x})$ be the characteristic function of the web, defined as the region exterior to all the boundary curves. In the sense of tempered distributions, does χ_{web} have a Fourier expansion with a countable discrete set of wave vectors?

Because of the invariance of M_q^q under 2π translations, it is easy to extend χ_{web} to a periodic function in $\mathbb{R}^q$. What is not at all easy is to show that the function thus obtained is Fourier analyzable. By our definition of the web, this is entirely a question about the nature of the boundary. As before, it is useful to slice up q-dimensional space into hyperplanes $S_\eta \equiv \eta + S$, $\eta \in S_\perp$, with S parametrized by x, y. In each S_η we define the web by means of the maximal simple, closed, invariant curves of $\tilde{M}_\eta^{(q)}$. The q-dimensional web is just the union of these planar webs.

Let us now make some plausible conjectures concerning the boundary set B_q of the q-dimensional stochastic web:

Conjecture 1. As a boundary curve between quasiperiodicity and chaos, the restriction of B_q to S_η is continuous, but not smooth; its derivative fails to exist at a countable dense set of points. Nevertheless, the Fourier transform of χ_{web} restricted to S_η exists as a tempered distribution.

Conjecture 2. As a function of $\eta \in S_\perp$, B_q is continuous almost everywhere, with discontinuities at a dense set of points. This roughness is a consequence of the fact that for a given η each boundary curve has a rotation number ρ that changes as one varies η. When ρ changes from one irrational value to another, say from α to β, all of the island chains corresponding to rotation numbers between α and β are either merged with the web or separated from it. With each merger or separation comes a discrete jump in the boundary curve. If the situation is not more complicated than this, the discontinuities will not prevent the Fourier transform from existing (the problem is analogous to the Fourier series expansion of a periodic one-dimensional devil's staircase with a dense set of discontinuities of bounded variation on any finite interval).

Conjecture 3. The restriction of the q-dimensional Fourier expansion to S exists, and hence the stochastic web is quasiperiodic, for all but a countable dense set of parameter values a. The exceptional parameter values are those for which the q-dimensional web is discontinuous at $\eta = 0$, i.e., for which one of the boundary curves is the unique boundary between the web and the stochastic layer of a neighboring island chain.

Obviously, the conjectured quasiperiodicity is unlikely to be proved or disproved very soon. But even without rigorous confirmation of the quasiperiodicity of the web, we are not left empty handed. Our general method still allows us to strengthen the concept of dynamical quasisymmetry generation. As mentioned at the outset, previous authors [2,3] have used this terminology to refer to the first-order (in a) approximation of the stochastic web to a strictly quasiperiodic level set of the Hamiltonian H_2. In Ref. [9] higher-order quasiperiodic Hamiltonians were introduced that allow an improved approximation to the web boundary. But replacing the web boundary in $\mathbb{R}^q$ piecewise by manifolds of constant energy is only one of many possible ways of smoothing the boundary sufficiently to guarantee Fourier analyzability in $\mathbb{R}^q$ and quasiperiodicity in the xy plane. Even for relatively large a, for which chaotic orbits dominate the phase portrait, it should still be possible to exploit the q-dimensional embedding technique to find a strictly quasiperiodic approximation to the boundary of the stochastic region.

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